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Heteroclinic Attractors: Time Averages and Moduli of Topological Conjugacy

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Abstract. In this paper we consider attracting heteroclinic cycles. We recall that these cycles usually have no S.B.R. measure. This is related with the fact that certain time averages do not converge. We obtain a topological interpretation of the asymptotic properties of these non-converging time averages. In terms of these asymptotic properties we obtain a complete set of moduli for the attracting heteroclinic cycles.

1. Introduction

In this paper we consider heteroclinic 'attractors' (they differ from usual attractors in the sense that they are only attracting from one side) as in an example by Bowen of a smooth (at least C^3) vectorfield in the plane with two saddle points and two saddle connections as indicated in figure 1. The eigenvalues of the linearized vector fields at the saddle points are supposed to be so that the cycle, consisting of the two saddle connections, is attracting from the inside. The special property of the example is that for no orbit, converging to this cycle, there is a corresponding 'physical measure'.

We say that μ is the physical measure corresponding to an orbit $x(t)$ of a flow in a space X if, for each continuous function $g: X \to \mathbb{R}$, the phase average $\int_x g d\mu$ equals the time average $\lim_{t\to\infty} 1/t \int_0^t g(x(s))ds$. In other words, the physical measure of an orbit describes the probability of finding a point of the orbit $x(t)$, for big values of t, in the different regions of the phase space X; see also [**R,1989**]. This notion of a physical measure is closely related to the notion of an S.B.R. measure, see [**S,1970**]. In particular in this example the loop of saddle connections is

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an 'attracting' set without such S.B.R. measure.

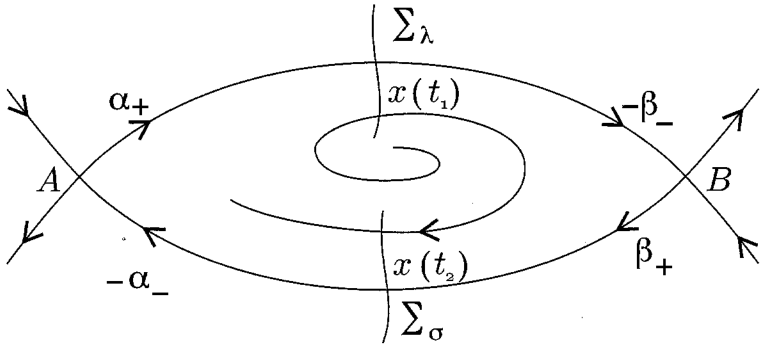

Figure 1: Phase portrait of the example by Bowen.

In the example by Bowen, if $x(t)$ is an integral curve, converging to the cycle, and if g is a continuous function on the plane, taking different values in the saddle points A and B, the time average

$$\lim_{t\to\infty}\int_0^t g(x(s))ds$$

does not exist. This means that in this example there is an open set of initial states (the basin of attraction of the cycle) such that the corresponding orbits define *non*-stationary time series (whenever one uses an observable which has different values in the two saddle points).

I attribute this example to Bowen: although he never published it, I learned this example through a paper by Ruelle who referred to Bowen (we have no reference: even Ruelle could not localize that paper). The example is ungeneric in the sense that it is not persistent under perturbations (which can break the saddle connections). In the mean time these homoclinic cycles were also discovered independently by other mathematicians in situations where they are persistent. In the context of population biology, were one uses dynamical systems on a simplex, they were studied by Gaunersdorfer [**G,1992**], who proved non-convergence of time averages (as was pointed out to me by M. Krupa). Also in the context of symmetric vector fields the homoclinic cycles can occur in a persistent way, and were analyzed e.g. in [**GH,1988**].

In [**G,1992**] there are relations between the *limsup* and the *liminf* of the partial time averages

$$\bar{g}(t) = 1/t \int_0^t g(x(s))ds$$

and the eigenvalues of the linearized flows at the saddle points. These results can in fact be formulated in terms of the moduli of topological conjugacy as introduced in [**P,1978**].

In this paper we extend the relations between the asymptotic behaviour of the partial time averages and the moduli of topological conjugacy. We construct a new modulus, the value of which can be determined from the asymptotic behaviour of the partial time averages (and hence is in invariant under topological conjugacy) and which, together with the moduli defined before (and whose values can also be determined from the asymptotic behaviour of the partial time averages), determine the heteroclinic attractor and its basin of attraction up to topological conjugacy. Our results admit straightforward generalizations to heteroclinic cycles with more than 2 saddles in dimension 2.

The non-existence of physical measures is against physical intuition. This intuition is supported by the ergodic theorem [**B,1931**] which implies that for dynamical systems preserving a probability measure, almost every (in the sense of that measure) orbit defines a physical measure. See also the survey on time averages by Sigmund [**S,1992**]. The example of Bowen, described above, has an open set of orbits which do not define physical measures, but the example itself is exceptional: it has two saddle connection which can be perturbed away. As we observed before, for dynamical systems with symmetry and for dynamical systems on a simplex, heteroclinic attractors occur in a persistent way. It is however an interesting and probably difficult question to determine whether there are such persistent examples for general dynamical systems (smooth vectorfields or maps on $\mathbb{R}^n$).

2. Statement of the main results

We denote, for the example of Bowen given in figure 1, the expanding and contracting eigenvalues of the linearized vector field in A by α_+ and

$-\alpha_-$ and in B by β_+ and $-\beta_-$ – we recall that the saddle points are denoted by A and B. The condition on the eigenvalues which makes the cycle attracting is that the contracting eigenvalues dominate: $\alpha_-\beta_- > \alpha_+\beta_+$.

The modulus associated with the upper, respectively lower, saddle connection is denoted by λ, respectively σ. They are defined by

$$\lambda = \alpha_-/\beta_+ \text{ and } \sigma = \beta_-/\alpha_+,$$

their values are positive and their product is bigger than 1, assuming the cycle to be attracting.

We first restate the result of Gaunersdorfer, for our present case of a two saddle cycle, in terms of the moduli of topological conjugacy:

Theorem 1. (See also [**G,1992**]). *If g is a continuous function on $\mathbb{R}^2$ with $g(A) > g(B)$, and $x(t)$ an orbit converging to the cycle, then we have for the partial time averages $\bar{g}(t)$:*

$$\limsup \bar{g} = \frac{\sigma}{1+\sigma} g(A) + \frac{1}{1+\sigma} g(B)$$
$$\liminf \bar{g} = \frac{\lambda}{1+\lambda} g(B) + \frac{1}{1+\lambda} g(A).$$

In order to formulate our main results we need to introduce some more notation. We continue to use $\bar{g}$ for the partial time averages of $g(x(t))$ as defined in the introduction. For big values of t, $x(t)$ is during most of the time near A or near B. To formalize this, let U_A and U_B be small neighbourhoods of A and B such that the function g is on these neighbourhoods bigger than the above limsup, respectively smaller than the above liminf (still assuming that $g(A) > g(B)$). Then for values of t for which $x(t)$ is in U_A, $\bar{g}$ is increasing, while for values of t for which $x(t)$ is in U_B, $\bar{g}$ is decreasing. The intervals for which $x(t)$ is in U_A or U_B are increasing in length unboundedly, the transition intervals, in which $x(t)$ is going from U_A to U_B, or vice versa, are uniformly bounded in length. Now we take sections Σ_λ and Σ_σ, as indicated in figure 1 intersecting the upper, respectively lower, saddle connection transversally. The successive times at which our orbit $x(t)$ intersects these sections are denoted by $t_1 < t_2$, etc., so that $x(t_1) \in \Sigma_\lambda$, $x(t_2) \in$

Σ_σ, etc. By taking the sections smaller, one may miss the first few intersections and get a sequence $\tilde{t}_i = t_{i+2k}$ for some integer k. Also, since we are only interested in the asymptotic behaviour of $\{t_i\}$ for $i \to \infty$, we can allow the numbering i to start at a value different from 1.

Theorem 2. *In the above situation the following limits exist:*

$$\lim \frac{t_{2i+1} - t_{2i}}{t_{2i} - t_{2i-1}} = \beta_-/\alpha_+ = \sigma$$
$$\lim \frac{t_{2i+2} - t_{2i+1}}{t_{2i+1} - t_{2i}} = \alpha_-/\beta_+ = \lambda.$$

This implies that

$$\lim \frac{t_{2i+2} - t2i}{t_{2i} - t_{2i-2}} = \lambda\sigma.$$

Also the limit

$$\lim t_{2i+2} - t_{2i} - \lambda\sigma(t_{2i} - t_{2i-2})$$

exists, and is a new invariant of topological conjugacy (or modulus) which we denote by ν.

Our final result is that the three moduli, mentioned up to now, are complete.

Theorem 3. *If we have two vector fields X and X' on the plane $\mathbb{R}^2$, both with a cycle as in the Bowen example, then there is a conjugacy of the closure of the domain of attraction of one of the closure of the domain of attraction of the other if and only if for X and X' the three moduli (here denoted by λ, σ, and ν) are equal.*

3. Proof of the theorems

3.1 Linearizations

It is known that, for saddle points of C^3 vector fields in $\mathbb{R}^2$, one can construct linearizing coordinates which are $C^{1+\varepsilon}$, where ε is a positive number, depending on the eigenvalues of the linear part of the vector field at the saddle point. Without loss of generality we assume in what follows that $\varepsilon < 1$. This was proved by Hartman [**H,1960**], even for vector fields which are C^1 and whose first derivative is Lipschits. In the present case we not only need these linearizing coordinates, but we also need a certain freedom in constructing them. For this reason we rather refer to **[PT,1993]**, where a more geometric construction is given from which it is clear what freedom one has in the choice of these linearizing coordinates, but where we need the vectorfield to be C^3. Below we describe these results in the form which we need here.

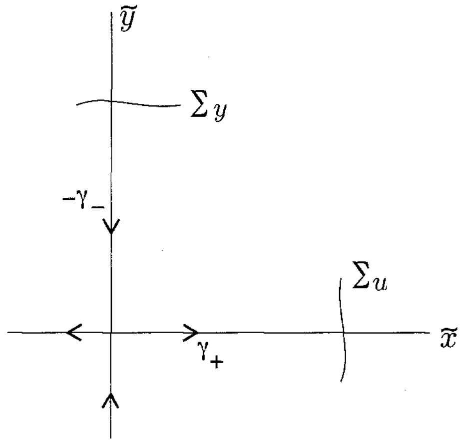

Figure 2: Linearizing coordinates near a saddle point in the plane.

We consider a C^3 vectorfield Z on the plane, which has a saddle point in the origin with contracting, respectively expanding, eigenvalue $-\gamma_-$, respectively γ_+; see figure 2. Then there are $C^{1+\varepsilon}$ coordinates $\tilde{x}$ and $\tilde{y}$, defined in a neighbourhood of the origin, such that, in these coordinates, the flow of Z has the form $Z_t(\tilde{x}, \tilde{y}) = (\tilde{x}\exp(\gamma_+ t),\ \tilde{y}\exp(-\gamma_- t))$. These coordinates are not unique. In fact, if one takes a (small) section Σ_y, transverse to the stable separatrix of the saddle, and a (small) section Σ_x, transverse to the unstable separatrix of the saddle (see figure 2), one can take these linearizing coordinates in such a way that $\tilde{y}$ and $\tilde{x}$ on Σ_y, respectively Σ_x, equal one. With this requirement the linearizing coordinates are even uniquely determined in the quadrant where they are both positive.

Changing the section Σ_y to $Z_t(\Sigma_y)$ (where Z_t denotes the flow of Z), corresponds to multiplication of $\tilde{y}$ with a factor $\exp(\gamma_- t)$; changing Σ_x

to $Z_t(\Sigma_x)$, corresponds to multiplication of $\tilde{x}$ with a factor $\exp(-\gamma_+ t)$.

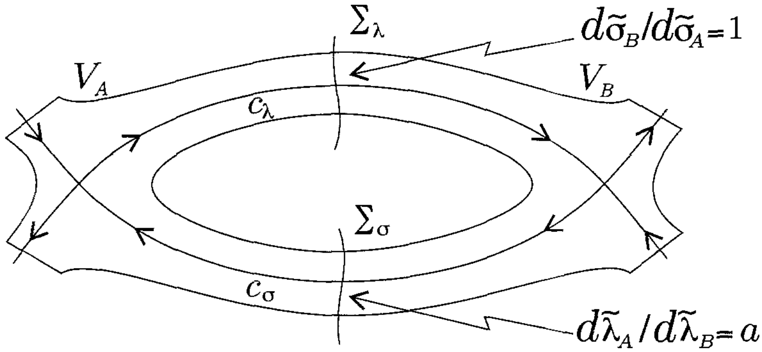

Figure 3: Domains V_A and V_B of the linearizing coordinates.

Now we apply this to the example of Bowen; see also figure 3. We take sections Σ_λ and Σ_σ transverse to the upper saddle connection c_λ, respectively the lower saddle connection c_σ as in the previous section. Then we take linearizing coordinates $\tilde{\lambda}_A$ and $\tilde{\sigma}_A$ in a neighbourhood V_A of A such that $\tilde{\lambda}_A$ and $\tilde{\sigma}_A$, restricted to Σ_λ, respectively Σ_σ, are equal to one. In the same way one constructs linearizing coordinates, denoted by $\tilde{\lambda}_B$ and $\tilde{\sigma}_B$, in a neighbourhood V_B of B which are, restricted to Σ_λ, respectively Σ_σ, equal to one.

On these sections Σ_A and Σ_B we have, apart from the linearizing functions which are equal to one, restrictions of linearizing functions which can be used as a coordinates (and which are zero in the intersection of the section with the saddle connection). On each of these sections we have two of these coordinate functions: one defined on V_A and one defined on V_B. In general these two functions are not equal on the section – this complicates the proofs and is related with the modulus ν. In general one can even not take the sections Σ_λ and Σ_σ so that the two coordinate functions on both sections coincide. The best we can do is to make on one of the sections, say on Σ_λ the first derivatives of the two coordinate functions equal in the point of intersection with the saddle connection c_λ. This can be done by replacing Σ_σ by $X_t(\Sigma_\sigma)$, for an appropriate value of t. In fact, doing so has the effect that $\tilde{\sigma}_A$ is multiplied by a factor $\exp(\alpha_- t)$ and $\tilde{\sigma}_B$ by a factor $\exp(-\beta_+ t)$. From this one easily sees that by moving the section Σ_σ, one can make the derivatives of the two coordinate functions on the other section Σ_λ equal in the point of intersection with the saddle connection. One cannot improve things by also moving the section Σ_λ, because if we apply X_t to both sections then essentially nothing changes: the coordinate functions will then just be composed with X_{-t}. The ratio of the derivatives of the two coordinate functions on Σ_λ in the intersection with c_λ is related with the modulus ν.

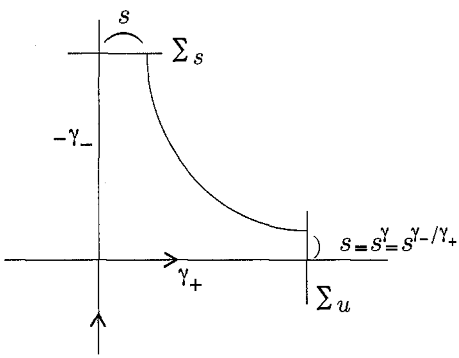

Figure 4: Transition map of a saddle point of a linear vector field.

3.2 Proof of the first part of theorem 2 and of theorem 1

We start with the first two limits in theorem 2. For this we recall some facts concerning linear vector fields on the plane.

Let Z be a linear vector field on the plane, given by $Z = \gamma_+\partial/\partial x - \gamma_-\partial/\partial y$, with both γ_+ and γ_- positive. The sections $\{x = 1\}$ and $\{y = 1\}$ are denoted by Σ_u and Σ_s; see figure 4. For a point $(s, 1)$ in Σ_s, with $s > 0$, the integration time t, needed to reach the other section Σ_u, is given by

$$t = -\frac{\ln s}{\gamma_+}.$$

The point of intersection of the orbit through $(s, 1)$ with Σ_u is given by

$$(1, s' = s^\gamma) \text{ where } \gamma = \gamma_-/\gamma_+.$$

We apply the above considerations to our example, using again the notation as shown in figure 3. We consider an orbit which first passes through V_B and then through V_A.

First in V_B, using the linearizing coordinates $\tilde{\lambda}_B$ and $\tilde{\sigma}_B$, we take the orbit entering at $\tilde{\sigma}_B = s$. This orbit will stay during time $-(1/\beta_+)\ln s$ in V_B and will exit at $\tilde{\lambda}_B = s^\beta$ (as above with γ we use $\beta = \beta_-/\beta_+$ and $\alpha = \alpha_-/\alpha_+$).

Then it enters in V_A at $\tilde{\lambda}_A = f(s^\beta) = a \cdot s^\beta + O(s^{\beta(1+\varepsilon)}) = s'$, where f is the function which describes the transition from the B-linearizing coordinate to the A-linearizing coordinate. The derivative of this function in 0 is denoted by a, and this function has the same differentiability as the linearizing coordinates, namely $C^{1+\varepsilon}$. The orbit will now stay in V_A during a time $-(1/\alpha_+)\ln s' = -(1/\alpha_+)(\ln a + \beta \ln s + O(s^{\beta\varepsilon}))$. Now we easily see that the quotient of these successive times (more precisely: the time spent in V_A divided by the time spent in V_B) approaches, as s tends to zero, the value $(\beta\beta_+)/\alpha_+ = \beta_-/\alpha_+ = \sigma$.

This proves the first limit in theorem 2. The second can be derived in the same way.

For later reference we notice that the above arguments give even the slightly stronger result that

$$\lim_{i\to\infty}(t_{2i+1} - t_{2i} - \sigma(t_{2i} - t_{2i-i})) = -(1/\alpha_+)\ln a.$$

and

$$\lim_{i\to\infty}(t_{2i+2} - t_{2i+1} - \lambda(t_{2i+1} - t_{2i})) = 0.$$

Next we give the proof of theorem 1. We only derive the limsup – the formula for the liminf can be obtained in the same way. We divide the orbit $x(t)$ in 'periods', each starting in Σ_λ (note that the different periods are *not* equally long). During these periods, the ratio between the time spent in V_A and V_B, approaches $\sigma{:}\,1$. Since, for increasing t the orbit spends an increasing part of the time it is in V_B in a very small neighbourhood of B (and the same for V_A), the partial time averages, at the *end* of the periods, will approach the weighted average of $g(A)$ and $g(B)$:

$$\frac{\sigma}{1+\sigma}g(A) + \frac{1}{1+\sigma}g(B).$$

Since the partial time averages are increasing as long as the orbit is

V_A and is decreasing as long as the orbit is in V_B (except if it is in the transition fase), the above value must be the limsup as stated in theorem 1. This completes the proof of the first theorem.

3.3 Proof of the second part of theorem 2

In this proof we prefer to divide the orbit $x(t)$ in periods, which begin and end each time the orbit is passing Σ_σ – so the periods are separated by the t values t_{2i}. First we consider two successive periods. We assume that the first period starts when the orbit enters V_A at a point where the $\tilde{\lambda}_A$ coordinate has the value s. By the same type of arguments as used in the previous subsection one derives that the total time of this cycle is

$$-(1/\alpha_+ + \alpha/\beta_+)\ln s + O(s^{\alpha\varepsilon})$$

and that the total time of the next cycle is

$$-(1/\alpha_+ + \alpha/\beta_+)(\alpha\beta \ln s + \ln a) + O(s^{\alpha\varepsilon}),$$

where a, as before denotes the derivative of $\tilde{\lambda}_A$ with respect to $\tilde{\lambda}_B$ at the intersection of Σ_σ with the separatrix c_σ and where the linearizing coordinates are $C^{1+\varepsilon}$.

This means that

$$t_{2i+2} - t_{2i} - \alpha\beta(t_{2i} - t_{2i-2}) = -(1/\alpha_+ + \alpha/\beta_+)\ln a + R_i,$$

where $R_i = O(s_i^{\alpha\varepsilon})$, s_i being the $\tilde{\lambda}_A$ coordinate of $x(t_{i-2})$. Since $s_{i+1} = a \cdot s_i^{\alpha\beta}(1 + O(s^{\alpha\beta}))$, R_i is converging to zero. Since $\alpha\beta = \lambda\gamma$ this proves the second part of theorem 2. For the third modulus we have

$$\nu = -(1/\alpha_+ + \alpha/\beta_+)\ln a.$$

In the next subsection we show that ν is an invariant under topological conjugacy.

For later reference we note two things. First, that R_i converges so fast to zero that the infinite sum $\sum_i i|R_i|$ converges. Second, that $-(1/\alpha_+)\ln a$ can be expressed in terms of the moduli: $-(1/\alpha_+)\ln a =$

$(1/\alpha_+)(1/\alpha_+ + \alpha/\beta_+)^{-1}\nu = (1 + (\alpha\alpha_+)/\beta_+)^{-1}\nu = (1 + \lambda)^{-1}\nu.$

3.4 The proof of theorem 3

It is known that λ and σ as defined before are invariants under topological conjugation, see [**P,1978**]. In order to show that also ν is a topological invariant we proceed as follows. Let $x(t)$ be an orbit as before which is mapped by a topological conjugacy to an orbit $x'(t)$. For both orbits we can construct the sequences $\{t_i\}$ and $\{t'_i\}$ as before – since topological conjugacies do not need to respect the sections, these sequences need not coincide but one can make them – by changing the indices i to $i + 2k$, for some k, in one of the sequences – so that the differences $t_i - t'_i$ are uniformly bounded. Theorem 2 then implies that also the values of ν and ν' are equal for both vector fields. Hence ν is invariant under topological conjugacy.

Next we prove that the three moduli are complete. For a given vector field X on the plane as in the Bowen example, we construct another such vector field X' such that the three moduli are the same, and such that for the second vector field X' we have linearizing coordinates such that (denoting corresponding objects for X' by the same symbols as we used for X, but now with a prime) $\tilde{\lambda}'_A$ and $\tilde{\lambda}'_B$, restricted to Σ'_λ are equal and such that $\tilde{\sigma}'_A$, restricted to Σ'_σ, is equal to a' times $\tilde{\sigma}'_B$. We have to show that, in these two cases the vector field, restricted to the domains of attraction of the cycles, are topologically conjugated.

We first observe that for X', due to the special requirements, a stronger versions of theorem 2 holds: For any orbit $x'(t)$ converging to the cycle and $\{t'_i\}$ as defined before, we have

$$t'_{2i+2} - t'_{2i+1} - \lambda(t'_{2i+1} - t'_{2i}) = 0,$$
$$t'_{2i+1} - t'_{2i} - \sigma(t'_{2i} - t'_{2i-1}) = -(1/\alpha'_+)\ln a' = (1 + \lambda)^{-1}\nu$$

and hence

$$t'_{2i+2} - t'_{2i} - \lambda\sigma(t'_{2i} - t'_{2i-2}) = \nu.$$

This means that for the lengths of the successive periods $T'_i = t'_{2i+2} - t'_{2i}$, we have $T'_{i+1} = \alpha\beta T'_i + \nu$. For any sequence $\{t'_i\}$, satisfying the above formulas there is exactly one corresponding orbit of X' and the sequence also determines the initial point $x'(0)$ of the orbit.

Now we consider an orbit $x(t)$ of X and the corresponding sequence $\{t_i\}$. From the above observations it follows that a conjugacy should map $x(t)$ to an X' orbit $x'(t)$ such that the sequences $\{t_i\}$ and $\{t'_i\}$ of these orbits can be taken so that $t_i - t'_i$ is uniformly bounded. However we need more: since $x(t_{2i})$ converges, for i going to infinity, to $\Sigma_\sigma \cap c_\sigma$, also $x'(t_{2i})$ should converge to a point of c'_σ – this means that $t_i - t'_i$ should converge to a constant. In fact we shall construct the sequence t'_i such that the above formulas are satisfied (and hence such that there is a unique corresponding X' orbit $x'(t)$) and such that $t_i - t'_i$ converges to zero. This means that the conjugacy from $x(t)$ to $x'(t)$ extends to a conjugacy from c_σ to c'_σ in such a way that $\Sigma_\sigma \cap c_\sigma$ is mapped to $\Sigma'_\sigma \cap c'_\sigma$. We mentioned in the proof of theorem 2 that $t_{2i+1} - t_{2i} - \sigma(t_{2i} - t_{2i-1})$ converges to $-(1/\alpha_+)\ln a = (1+\lambda)^{-1}\nu = -(1/\alpha'_+)\ln a'$. This means that the conjugacy from $x(t)$ to $x'(t)$ also extends continuously to a conjugacy from c_λ to c'_λ mapping $\Sigma_\lambda \cap c_\lambda$ to $\Sigma'_\lambda \cap c'_\lambda$.

So to complete the construction of the conjugacy, restricted to the orbit $x(t)$ (and its closure), we only have to construct the sequence $\{t'_i\}$ satisfying $\lim_{i\to\infty} t'_i - t_i = 0$ and satisfying

$$t'_{2i+2} - t'_{2i} - \lambda\sigma(t'_{2i} - t'_{2i-2}) = \nu,$$

and

$$t'_{2i+2} - t'_{2i+1} - \lambda(t'_{2i+1} - t'_{2i}).$$

We first concentrate on the conditions involving t_i and t'_i for even values of i. We define $T_i = t_{2i+2} - t_{2i}$ and recall that $T_i - \lambda\sigma T_{i-1} = \nu + R_i$, with $\Sigma_i i|R_i|$ converging to a finite value. For each i, we construct a sequence $T_0^{(i)}, T_i^{(1)}, \ldots, T_i^{(i)} = T_i$ such that $T_j^{(i)} - \lambda\sigma T_{j-1}^{(i)} = \nu$ for $j = 1, \ldots, i$. Then $T_0^{(i+1)} - T_0^{(i)} = (\lambda\sigma)^{-(i+1)} R_{i+1}$, so $\lim_{i\to\infty} T_0^{(i)} = \bar{T}_0$ exists. We take this $\bar{T}_0$ as starting point of a sequence $\{\bar{T}_i\}$ such that $\bar{T}_i - \lambda\sigma\bar{T}_{i-1} = \nu$. Then we have $\lim(T_i - \bar{T}_i) = 0$ and $|T_i - \bar{T}_i| \le \Sigma_{j=i+1}^{\infty} |R_j|$. This implies that $\lim_{l\to\infty} \Sigma_{i=0}^{l}(T_i - \bar{T}_i)$ converges to a finite value. This means that we can take t'_{2i} so that $\bar{T}_i = t'_{2i+2} - t'_{2i}$ and $\lim(t_{2i} - t'_{2i}) = 0$.

Finally we take t'_{2i+1} so that $t'_{2i+2}-t'_{2i+1} = \lambda(t'_{2i+1}-t'_{2i})$. This concludes the construction of the sequence $\{t'_i\}$.

We obtain a topological conjugacy by applying the above construction to all orbits. It is easy to see that in this way we get a continuous map h such that $X'_t h = hX_t$. We only have to verify injectivity. This follows from the fact that for two different orbits $x(t)$ and $\bar{x}(t)$ of X, the asymptotic behaviour of the corresponding sequences $\{t_i\}$ and $\{\bar{t}_i\}$ are not the same. This last fact can be seen as follows.

Without loss of generality e may assume that $t_0 = \bar{t}_0 = 0$, so that $x(0)$, $\bar{x}(0) \in \Sigma_\sigma$, and also that the $\tilde{\lambda}_A$ coordinates of both points are sufficiently small. Then it follows that the ratio of the $\tilde{\lambda}_A$ coordinates s_i and $\bar{s}_i$, of the intersections $x(t_{2i})$ and $\bar{x}(t_{2i})$ of the orbits with Σ_σ, goes to zero or to infinity. On the other hand, as we have seen, the values of $t_{2i+2} - t_{2i}$ and $\bar{t}_{2i+2} - \bar{t}_{2i}$ are asymptotically, up to a fixed multiplicative and a fixed additive constant, equal to $\ln s_i$, respectively $\ln \bar{s}_i$. This implies that $t_i - \bar{t}_i$ cannot be uniformly bounded. This completes the proof of theorem 3.

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